The quantum state cannot be interpreted statistically

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Quantum states are the key mathematical objects in quantum theory. It is therefore surprising that physicists have been unable to agree on what a quantum state represents. There are at least two opposing schools of thought, each almost as old as quantum theory itself. One is that a pure state is a physical property of system, much like position and momentum in classical mechanics. Another is that even a pure state has only a statistical significance, akin to a probability distribution in statistical mechanics. Here we show that, given only very mild assumptions, the statistical interpretation of the quantum state is inconsistent with the predictions of quantum theory. This result holds even in the presence of small amounts of experimental noise, and is therefore amenable to experimental test using present or near-future technology. If the predictions of quantum theory are confirmed, such a test would show that distinct quantum states must correspond to physically distinct states of reality.

A quantum wave function was originally conceived by Schröedinger as a tangible, physical wave. This viewpoint was quickly threatened both by Born relating the wave function to probabilities, and by the realisation that quantum states could not always be assigned separately to individual systems. Nevertheless most physicists and chemists concerned with pragmatic applications successfully treat the quantum state as a real object encoding all properties of microscopic systems.

However, many [1–8] have suggested that the quantum state should properly be viewed as something less than real. For example:

> *... I incline to the opinion that the wave function does not (completely) describe what is real, but only a (to us) empirically accessible maximal knowledge regarding that which really exists [...] This is what I mean when I advance the view that quantum mechanics gives an incomplete description of the real state of affairs.* –A. Einstein [9]

The motivation for physicists to take an interest in this question was eloquently stated by Jaynes:

> *But our present QM formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature — all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble. Yet we think that the unscrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about; it is just that simple.* –E. T. Jaynes [10]

Some physicists hold that quantum systems do not have physical properties, or that the existence of quantum systems at all is a convenient fiction. In this case, the state vector is a mere calculational device, used to make predictions of the probabilities for macroscopic events. This work, however, proceeds on the assumption that quantum systems – like atoms and photons – exist, and have at least some physical properties. We assume very little about these properties, for example we do not assume that systems have a definite position or momentum. The statistical view of the quantum state is that it merely encodes an experimenter's information about the properties of a system. We will describe a particular measurement and show that the quantum predictions for this measurement are incompatible with this view.

We begin by describing more fully the difference between the two different views of the quantum state [11]. Consider two different methods of preparing a quantum system. If method 1 is used, quantum theory assigns a pure state $|\phi_0\rangle$. If method 2 is used, quantum theory assigns a pure state $|\phi_1\rangle \neq |\phi_0\rangle$, where $|\phi_0\rangle$ and $|\phi_1\rangle$ are non-orthogonal. Our main assumption is that after preparation, the quantum system has some set of physical properties. These may be completely described by quantum theory, but in order to be as general as possible, we allow that they are described by some other, perhaps undiscovered theory. Assume that a complete list of these physical properties corresponds to some mathematical object, λ.

If the quantum state is a physical property of the system (the first view), then either λ is identical with $|\phi_0\rangle$ or $|\phi_1\rangle$, or λ consists of $|\phi_0\rangle$ or $|\phi_1\rangle$, supplemented with values for additional variables not described by quantum theory. Either way, the quantum state is uniquely determined by λ.

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If the quantum state is statistical in nature (the second view), then a full specification of λ need not determine the quantum state uniquely. Some values of λ may be compatible with the quantum state being either $|\phi_0\rangle$ or $|\phi_1\rangle$. This can be understood via a classical analogy. Suppose there are two different methods of flipping a coin, each of which is biased. Method 1 gives heads with probability $p_0 > 0$ and method 2 with probability $0 < p_1 \neq p_0$. If the coin is flipped only once, there is no way to determine by observing only the coin which method was used. The outcome heads is compatible with both. The statistical view says something similar about the quantum system after preparation. The preparation method determines either $|\phi_0\rangle$ or $|\phi_1\rangle$ just as the flipping method determines probabilities for the coin. But a complete list of physical properties λ is analogous to a list of coin properties, such as position, momentum, etc. Just as "heads up" is compatible with either flipping method, a particular value of λ might be compatible with either preparation method.

We will show that the statistical view is not compatible with the predictions of quantum theory. We will begin with a simple version of the argument, which works when $|\langle\phi_0|\phi_1\rangle| = 1/\sqrt{2}$. Then the argument is extended to arbitrary $|\phi_0\rangle$ and $|\phi_1\rangle$. Finally, we present a version of the argument which works even in the presence of experimental error and noise.

The simple argument is as follows. Choose a basis of the Hilbert space so that $|\phi_0\rangle = |0\rangle$ and $|\phi_1\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. In order to derive a contradiction, assume that there is some chance that the complete physical state, λ, of the system is compatible with either preparation method. Suppose that for either method, the probability of this happening is at least $q > 0$. (Of course it may be the case that, given λ, one method is more likely than the other – the only assumption here is that some fraction q of the time, λ does not determine *with certainty* which method was used.)

Now consider two systems whose physical states are uncorrelated. This can be achieved, for example, by constructing and operating two copies of a preparation device independently. The experiment is illustrated in Figure 1. With probability $q^2 > 0$ it happens that the physical states λ_1 and λ_2 are compatible with any of the four possible preparations. This means that the physical state of the two systems is compatible with any of the four possible quantum states $|0\rangle\otimes|0\rangle$, $|0\rangle\otimes|+\rangle$, $|+\rangle\otimes|0\rangle$ and $|+\rangle\otimes|+\rangle$.

The two systems are brought together and measured. An important assumption for the argument now is that the behaviour of the measuring device – in particular the probabilities for different outcomes – is only determined by the complete physical state of the two systems at the time of measurement, along with the physical properties of the measuring device. Once these things are specified, there can be no remaining dependence on the quantum state of the two systems.

The measurement is a joint measurement of the two systems, which projects onto the four orthogonal states:

$$
\begin{aligned}
|\xi_1\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle),\\
|\xi_2\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle\otimes|-\rangle + |1\rangle\otimes|+\rangle),\\
|\xi_3\rangle &= \tfrac{1}{\sqrt{2}}(|+\rangle\otimes|1\rangle + |-\rangle\otimes|0\rangle),\\
|\xi_4\rangle &= \tfrac{1}{\sqrt{2}}(|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle),
\end{aligned}
\qquad (1)
$$

where $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. The first outcome is orthogonal to $|0\rangle\otimes|0\rangle$, hence quantum theory predicts that this outcome has probability zero when the quantum state is $|0\rangle\otimes|0\rangle$. Similarly the second outcome has probability zero if the state is $|0\rangle\otimes|+\rangle$, the third if $|+\rangle\otimes|0\rangle$, and the fourth if $|+\rangle\otimes|+\rangle$. This leads immediately to the desired contradiction. At least q^2 of the time, the measuring device is uncertain which of the four possible preparation methods was used, and on these occasions it runs the risk of giving an outcome that quantum theory predicts should occur with probability 0. Importantly, we have needed to say nothing about the value of q *per se* to arrive at this contradiction.

This argument shows that no physical state λ of the system can be compatible with both of the quantum states $|0\rangle$ and $|+\rangle$. If the same can be shown for any pair of quantum states $|\phi_0\rangle$ and $|\phi_1\rangle$, then the quantum state can be inferred uniquely from λ. In this case, the quantum state is a physical property of the system, and the statistical view is false.

For any pair of distinct non-orthogonal states $|\phi_0\rangle$ and $|\phi_1\rangle$, a basis of the Hilbert space can be chosen such that

$$
\begin{aligned}
|\phi_0\rangle &= \cos(\theta/2)\,|0\rangle - \sin(\theta/2)\,|1\rangle\\
|\phi_1\rangle &= \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle\,,
\end{aligned}
\qquad (2)
$$

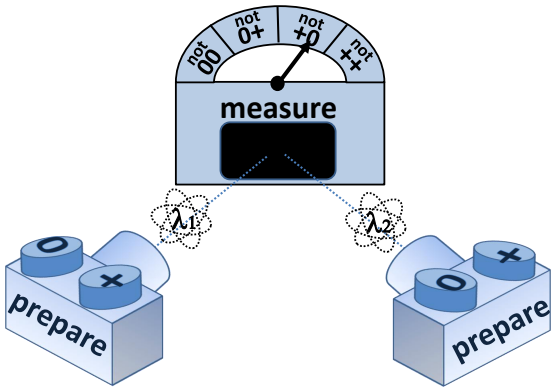

FIG. 1. Two systems are prepared independently. The quantum state of each, determined by the preparation method, is either $|0\rangle$ or $|+\rangle$. The two systems are brought together and measured. The outcome of the measurement can only depend on the physical properties of the two systems at the time of measurement.

with $0 < \theta < \pi/2$. These two states span a two-dimensional subspace of the Hilbert space. We can restrict attention to this subspace and from hereon, without loss of generality, treat the systems as qubits. Assume, as above, that there is a probability at least $q > 0$ that the complete physical state of the system after preparation is compatible with either preparation method having been used.

A contradiction is obtained when n uncorrelated systems are prepared, where n will be fixed shortly. Depending on which of the two preparation methods is used each time, the n systems are prepared in one of the quantum states:

$$\begin{aligned}
|\Psi(0\ldots00)\rangle &= |\phi_0\rangle \otimes \cdots \otimes |\phi_0\rangle \otimes |\phi_0\rangle \\
|\Psi(0\ldots01)\rangle &= |\phi_0\rangle \otimes \cdots \otimes |\phi_0\rangle \otimes |\phi_1\rangle \\
&\vdots \\
|\Psi(x_1\ldots x_n)\rangle &= |\phi_{x_1}\rangle \otimes \cdots \otimes |\phi_{x_{n-1}}\rangle \otimes |\phi_{x_n}\rangle \\
&\vdots \\
|\Psi(1\ldots1)\rangle &= |\phi_1\rangle \otimes \cdots \otimes |\phi_1\rangle \otimes |\phi_1\rangle \,. \qquad (3)
\end{aligned}$$

Since these preparations are independent, there is a probability at least q^n that the complete physical state of the systems emerging from the devices is compatible with any one of these 2^n quantum states. The desired contradiction is obtained if there is a joint measurement on the n systems with 2^n outcomes such that each outcome has probability zero on at least one of the $|\Psi(x_1\ldots x_n)\rangle$ [12].

A suitable measurement is most easily described as a quantum circuit, followed by a measurement onto the $\{|0\rangle, |1\rangle\}$ basis for each qubit. It is illustrated in Figure 2.

The circuit is parameterized by two real numbers, α and β. In Appendix A it is shown that for any $0 < \theta < \pi/2$, and for any n chosen large enough that $2\arctan\left(2^{1/n}-1\right) \leq \theta$, it is possible to choose α and β such that the measurement has the desired feature: each outcome has, according to quantum theory, probability zero on one of the states $|\Psi(x_1\ldots x_n)\rangle$.

Finally, the argument so far uses the fact that quantum probabilities are sometimes exactly zero. The argument has not taken any account of the experimental errors that will occur in any real laboratory. It is very important to have a version of the argument which is robust against small amounts of noise. Otherwise the conclusion – that the quantum state is a physical property of a quantum system – would be an artificial feature of the exact theory, but irrelevant to the real world. Experimental test would be impossible.

For the noise-tolerant version of the argument, let Λ be a (measure) space of possible values that λ can take, for a given type of system. Just as a particular flipping method defines a probability distribution over the coin states *heads* and *tails*, a particular preparation method defines a probability distribution on Λ. Let the distribution be $\mu_i(\lambda)$ when the quantum state $|\phi_i\rangle$ is prepared.

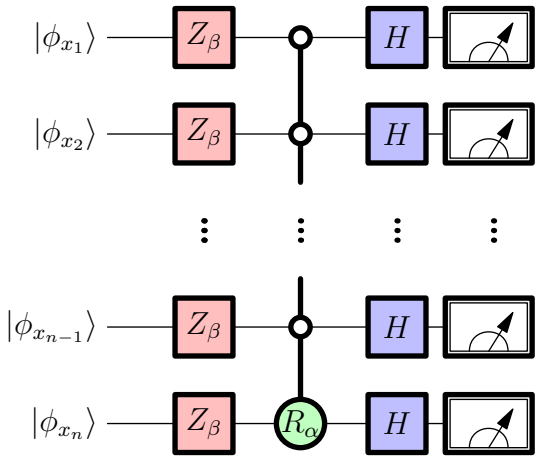

FIG. 2. The main argument requires a joint measurement on n qubits with the property that each outcome has probability zero on one of the input states. Such a measurement can be performed by implementing the quantum circuit shown, followed by a measurement of each qubit in the computational basis. The single qubit gates are given by $Z_\beta = |0\rangle\langle 0| + e^{i\beta}|1\rangle\langle 1|$ and the Hadamard gate $H = |+\rangle\langle 0| + |-\rangle\langle 1|$. The entangling gate in the middle rotates the phase of only one state: $R_\alpha|00\ldots0\rangle = e^{i\alpha}|00\ldots0\rangle$, leaving all other computational basis states unaffected.

Suppose that the above experiment is performed, with n systems prepared independently and the measurement of Figure 2 performed. In a real experiment, it will be possible to establish with high confidence that the probability for each measurement outcome is within ϵ of the predicted quantum probability for some small $\epsilon > 0$.

The noise tolerant result relates ϵ to the *total variation distance* [13] between μ_0 and μ_1, where the total variation distance is defined by

$$D(\mu_0, \mu_1) = \frac{1}{2}\int_\Lambda |\mu_0(\lambda) - \mu_1(\lambda)| \mathrm{d}\lambda. \qquad (4)$$

It is a measure of how easy it is to distinguish two probability distributions. If $D(\mu_0, \mu_1) = 1$, then μ_0 and μ_1 are completely disjoint. In this case, the probability of λ being compatible with both preparations (q above) is zero. In Appendix B we show that

$$D(\mu_0, \mu_1) \geq 1 - 2\sqrt[n]{\epsilon}. \qquad (5)$$

For small ϵ, $D(\mu_0, \mu_1)$ is close to 1. Hence a successful experiment would show that each λ is associated almost exclusively with only one of the two quantum states.

Performing an experiment to implement the circuit in Figure 2 for small values of n is challenging but not unrealistic given current technology. While all the gates required have already been demonstrated at some point, our result requires such gates acting with high fidelity in a non post-selected fashion (this latter because otherwise the measuring device can use the extra freedom in the postselection to escape the zero-probability outcomes those times it is unsure of the preparation procedure).

It may be useful to summarize the assumptions that are necessary for the result. Three can be identified. The first is that if a quantum system is prepared in isolation from the rest of the universe, such that quantum theory assigns a pure state, then after preparation the system has a well defined set of physical properties. This assumption is necessary for the question we address to make sense: if such physical properties don't exist, it is meaningless to ask whether or not the quantum state is among them. Note that while there are well-known obstacles to identifying physical properties of individual systems when they are entangled with other systems, these problems do not arise here, since each system is in a pure state.

The second assumption is that it is possible to prepare multiple systems such that their physical properties are uncorrelated. Experimentalists aim to achieve this either by building and operating different copies of the same experimental apparatus, or by reusing the same apparatus after a sufficient time period has elapsed that they are confident the later run is independent of the earlier.

The third assumption is that measuring devices respond solely to the physical properties of the systems they measure. We do not assume underlying determinism. Even given a full specification of λ, it may only be possible to make probabilistic predictions about the outcome of a measurement.

We conclude by outlining some consequences of the result. First, one motivation for the statistical view is the obvious parallel between the quantum process of instantaneous wave function collapse, and the (entirely non-mysterious) classical procedure of updating a probability distribution when new information is acquired. If the quantum state is a physical property of a system – as it must be if one accepts the assumptions above – then the quantum collapse must correspond to a real physical process. This is especially mysterious when two entangled systems are at separate locations, and measurement of one leads to an instantaneous collapse of the quantum state of the other.

In some versions of quantum theory, on the other hand, there is no collapse of the quantum state. In this case, after a measurement takes place, the joint quantum state of the system and measuring apparatus will contain a component corresponding to each possible macroscopic measurement outcome. This is unproblematic if the quantum state merely reflects a lack of information about which outcome occurred. But if the quantum state is a physical property of the system and apparatus, it is hard to avoid the conclusion that each marcoscopically different component has a direct counterpart in reality.

On a related, but more abstract note, the quantum state has the striking property of being an exponentially complicated object. Specifically, the number of real parameters needed to specify a quantum state is exponential in the number of systems n. This has a consequence for classical simulation of quantum systems. If a simulation is constrained by our assumptions – that is, if it must store in memory a state for a quantum system, with independent preparations assigned uncorrelated states – then it will need an amount of memory which is exponential in the number of quantum systems.

For these reasons and others, many will continue to hold that the quantum state is not a real object. We have shown that this is only possible if one or more of the assumptions above is dropped. More radical approaches [14] are careful to avoid associating quantum systems with any physical properties at all. The alternative is to seek physically well motivated reasons why the other two assumptions might fail.

ACKNOWLEDGMENTS

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Appendix A: The measurement circuit

Consider a preparation device which can produce a quantum system in either the state $|\phi_0\rangle$, or the state $|\phi_1\rangle$. Suppose that n copies of this device are used independently. Then there are 2^n possible joint states of the n systems, depending on whether $|\phi_0\rangle$ or $|\phi_1\rangle$ was prepared each time. This section shows that for any distinct $|\phi_0\rangle$ and $|\phi_1\rangle$, if the number of systems n is large enough, then there is a joint measurement with the following property: each outcome has zero probability given one of the preparations. As illustrated in Figure 2, the measurement can be implemented by a quantum circuit followed by a computational basis measurement.

Choose a basis $\{|0\rangle, |1\rangle\}$ such that

$$|\phi_0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle, \tag{A1}$$

$$|\phi_1\rangle = \cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle, \tag{A2}$$

where $|\langle\phi_0|\phi_1\rangle|^2 = \cos^2(\theta)$. By restricting attention to the subspace spanned by $|\phi_0\rangle$ and $|\phi_1\rangle$, we can without loss of generality take the quantum systems to be qubits. For reasons seen below, choose n large enough that

$$2\arctan\left(2^{1/n} - 1\right) \leq \theta. \tag{A3}$$

The circuit consists of a unitary rotation Z_β applied to each qubit, followed by an entangling gate R_α, followed by a Hadamard gate applied to each qubit. The initial rotation is given by

$$Z_\beta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}. \tag{A4}$$

The n-qubit gate R_α is defined via its action on the computational basis states. Let $R_\alpha|0\cdots0\rangle = e^{i\alpha}|0\cdots0\rangle$,

and let R_α act as the identity on all other computational basis states. Finally, the Hadamard gate corresponds to the unitary operation

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{A5}$$

The action of the circuit is given by $U_{\alpha,\beta} = H^{\otimes n} R_\alpha {Z_\beta}^{\otimes n}$.

The measurement procedure consists of the unitary evolution $U_{\alpha,\beta}$ (for a particular choice of α and β discussed below), followed by a measurement of each qubit in the $\{|0\rangle, |1\rangle\}$ basis .

Let x_i be 0 (1) if the ith system is prepared in the state $|\phi_0\rangle$ ($|\phi_1\rangle$), and write $\vec{x} = (x_1, \ldots, x_n)$. Before the circuit is applied, the joint state of the n systems is

$$|\Psi(\vec{x})\rangle = |\phi_{x_1}\rangle \otimes \cdots \otimes |\phi_{x_n}\rangle. \tag{A6}$$

Hence the probability of the measurement outcome corresponding to the basis state $|x_1 \ldots x_n\rangle$ is the squared absolute value of

$$\begin{aligned}
&\langle x_1 \ldots x_n | H^{\otimes n} R_\alpha Z_\beta^{\otimes n} |\phi_{x_1}\rangle \otimes \cdots \otimes |\phi_{x_n}\rangle \\
&= \left(\sum_{\vec{z}} (-1)^{\vec{x}.\vec{z}} \langle \vec{z}| \right) R_\alpha Z_\beta^{\otimes n} |\phi_{x_1}\rangle \otimes \cdots \otimes |\phi_{x_n}\rangle \\
&= \left(e^{i\alpha} \langle 0 \cdots 0| + \sum_{\vec{z} \neq 00\cdots 0} (-1)^{\vec{x}.\vec{z}} \langle \vec{z}| \right) Z_\beta^{\otimes n} |\phi_{x_1}\rangle \otimes \cdots \otimes |\phi_{x_n}\rangle \\
&= \left(e^{i\alpha} \langle 0 \cdots 0| + \sum_{\vec{z} \neq 00\cdots 0} (-1)^{\vec{x}.\vec{z}} \langle \vec{z}| \right) \bigotimes_{i=1}^{n} \left(\cos\frac{\theta}{2} |0\rangle + (-1)^{x_i} e^{i\beta} \sin\frac{\theta}{2} |1\rangle \right) \\
&= \cos^n\frac{\theta}{2} e^{i\alpha} + \sum_{\vec{z} \neq 00\cdots 0} (-1)^{\vec{x}.\vec{z}} \left(\cos\frac{\theta}{2}\right)^{n-|\vec{z}|} \left(\sin\frac{\theta}{2}\right)^{|\vec{z}|} e^{i|\vec{z}|\beta} (-1)^{\vec{x}.\vec{z}} \\
&= \cos^n\frac{\theta}{2} e^{i\alpha} + \sum_{k=1}^{n} \binom{n}{k} \left(\cos\frac{\theta}{2}\right)^{n-k} \left(\sin\frac{\theta}{2}\right)^{k} e^{ik\beta} \\
&= \cos^n\frac{\theta}{2} \left(e^{i\alpha} + \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n - 1 \right).
\end{aligned} \tag{A7}$$

In the fifth line, $|\vec{z}| = \sum_i z_i$.

Finally, we show that for any θ with $2\arctan\left(2^{\frac{1}{n}} - 1\right) \leq \theta \leq \frac{\pi}{2}$, the angles α and β can be chosen so that

$$e^{i\alpha} + \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n - 1 = 0, \tag{A8}$$

and hence the probability is zero as required. Rearranging, the required α will always exist (and be easy to find) provided there exists a β with

$$\left| 1 - \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n \right| = 1. \tag{A9}$$

Such a β exists if the curve of $f(\beta) = 1 - \left(1 + e^{i\beta} \tan\frac{\theta}{2}\right)^n$ in the complex plane intersects the unit circle, as in Figure 3. Since f is continuous, it suffices to exhibit one point outside the unit circle and one point within it. Consider

$$f(0) = 1 - \left(1 + \tan\frac{\theta}{2}\right)^n. \tag{A10}$$

Since $\tan\frac{\theta}{2} \geq 2^{\frac{1}{n}} - 1$, $f(0) \leq -1$, hence it is outside (or on) the unit circle. On the other hand,

$$f(\pi) = 1 - \left(1 - \tan\frac{\theta}{2}\right)^n. \tag{A11}$$

Since $0 \leq \tan\frac{\theta}{2} \leq 1$, $0 \leq f(\pi) \leq 1$, hence it is inside (or on) the unit circle. This concludes the proof.

If the actual value of β for a particular θ and n is required, it is not difficult to find it numerically. For $n = 2$, (A9) can even be solved analytically to find $\beta = \arccos\left((1 - 4t^2 - t^4)/4t^3\right)$ where $t = \tan\frac{\theta}{2}$.

Appendix B: Noise-tolerant version of the argument

This section proves Eq. (5). This is a lower bound on the total variation distance between probability distributions corresponding to distinct quantum states, which holds even in the presence of noise.

Consider two methods of preparing a quantum system, such that quantum theory assigns the pure state $|\phi_0\rangle$ or

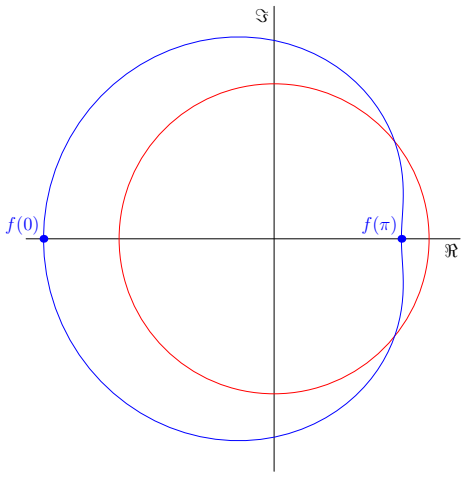

FIG. 3. Graph of $f(\beta)$ (blue), with $n = 2$ and $\theta = \frac{\pi}{3}$, and the unit circle (red). Suitable values for the parameters α and β exist if the curves intersect.

$|\phi_1\rangle$. An assumption is that the quantum system after preparation has physical properties, and that a complete list of these properties corresponds to some mathematical object λ. Each preparation method is associated with a probability distribution $\mu_i(\lambda)$ $(i = 0, 1)$. This is to be thought of as the probability density for the system to have properties λ after preparation. Another assumption is that when a measurement is performed, the behaviour of the measurement device depends only on the physical properties of the system and measuring device at the time of measurement. Formally, for a given measurement procedure M, the probability of outcome k is given by $P(k|M, \lambda) = \xi_{M,k}(\lambda)$, where $\xi_{M,k}$ is a function $\xi_{M,k} : \Lambda \to [0, 1]$. A model of this form reproduces the predictions of quantum theory exactly if

$$\int_\Lambda \xi_{M,k}(\lambda)\mu_i(\lambda)d\lambda = \langle\phi_i| E_{M,k} |\phi_i\rangle , \tag{B1}$$

where $E_{M,k}$ is the positive operator which quantum theory assigns to outcome k.

The total variation distance between the distributions $\mu_0(\lambda)$ and $\mu_1(\lambda)$ is

$$D(\mu_0, \mu_1) = \frac{1}{2}\int_\Lambda |\mu_0(\lambda) - \mu_1(\lambda)| \, d\lambda.$$

The aim is to show that if a model of the above form reproduces the predictions of quantum theory approximately, so that for any measurement outcome, Eq. (B1) holds to within ϵ, then

$$D(\mu_0, \mu_1) \geq 1 - 2\sqrt[n]{\epsilon}. \tag{B2}$$

Eq. (B2) holds for preparations of any pair of pure states $|\phi_0\rangle$ and $|\phi_1\rangle$, as long as n is chosen to satisfy Eq. (A3).

To this end, consider n independent preparations of quantum systems, where each can be chosen such that the quantum state is either $|\phi_0\rangle$ or $|\phi_1\rangle$. The joint quantum state is a direct product given by Eq. (A6). These systems will be brought together so that the joint measurement illustrated in Figure 2 and described in Appendix A can be performed.

We assume that the behaviour of the measurement device is determined by its own properties, and by a complete list $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of the physical properties for each one of the n systems. Seeing as the systems are prepared independently, the probability distribution for $\vec{\lambda}$ is given by

$$\mu_{\vec{x}}(\vec{\lambda}) = \mu_{x_1}(\lambda_1) \times \cdots \times \mu_{x_n}(\lambda_n). \tag{B3}$$

In order to prove Eq. (B2), it is useful to define a quantity which we call the *overlap* between $\mu_0(\lambda)$ and $\mu_1(\lambda)$:

$$\omega(\mu_0, \mu_1) = \int_\Lambda \min\{\mu_0(\lambda), \mu_1(\lambda)\}d\lambda. \tag{B4}$$

Note that $\omega(\mu_0, \mu_1) = 1 - D(\mu_0, \mu_1)$.

For probability distributions $\mu_1, \ldots, \mu_k$, the overlap can be generalised:

$$\omega(\mu_1, \ldots, \mu_k) = \int_\Lambda \min_i \mu_i(\lambda)d\lambda. \tag{B5}$$

Let Λ^n denote the n-fold Cartesian product of Λ, i.e. Λ^n is the space of possible values for $\vec{\lambda}$. From Eq. (B3),

$$\min_{\vec{x}} \mu_{\vec{x}}(\lambda_1, \ldots, \lambda_n) = \min\{\mu_0(\lambda_1), \mu_1(\lambda_1)\} \times \cdots \times \min\{\mu_0(\lambda_n), \mu_1(\lambda_n)\}. \tag{B6}$$

Integrating both sides gives

$$\omega\left(\{\mu_{\vec{x}}\}\right) = \int_{\Lambda^n} \min_{\vec{x}} \ \mu_{\vec{x}}(\vec{\lambda}) \ \mathrm{d}\vec{\lambda} = \left(\omega(\mu_0, \mu_1)\right)^n . \tag{B7}$$

Now if the initial state is $|\Psi(\vec{x})\rangle$, and the measurement of Figure 2 is performed, Appendix A shows that the outcome corresponding to the basis state $|\vec{x}\rangle$ has probability zero according to quantum theory. If a model of the above form assigns probability $\leq \epsilon$ to this outcome, for any $\vec{x}$, then

$$\int_{\Lambda^n} \xi_{M,\vec{x}}(\vec{\lambda})\mu_{\vec{x}}(\vec{\lambda})\mathrm{d}\vec{\lambda} \leq \epsilon. \tag{B8}$$

Since $\min_{\vec{x}} \mu_{\vec{x}}(\vec{\lambda}) \leq \mu_{\vec{x}}(\vec{\lambda})$, and both $\xi_{M,\vec{x}}(\vec{\lambda})$ and $\mu_{\vec{x}}(\vec{\lambda})$ are non-negative,

$$\int_{\Lambda^n} \xi_{M,\vec{x}}(\vec{\lambda}) \min_{\vec{x}} \mu_{\vec{x}}(\vec{\lambda})\mathrm{d}\vec{\lambda} \leq \epsilon. \tag{B9}$$

Finally, sum over $\vec{x}$ and use the normalization $\sum_{\vec{x}} \xi_{M,\vec{x}}(\vec{\lambda}) = 1$ to obtain

$$\omega\left(\{\mu_{\vec{x}}\}\right) \leq 2^n\epsilon. \tag{B10}$$

Combining Eqs. (B7) and (B10) gives

$$\left(\omega(\mu_0, \mu_1)\right)^n \leq 2^n\epsilon, \tag{B11}$$

which gives Eq. (B2).

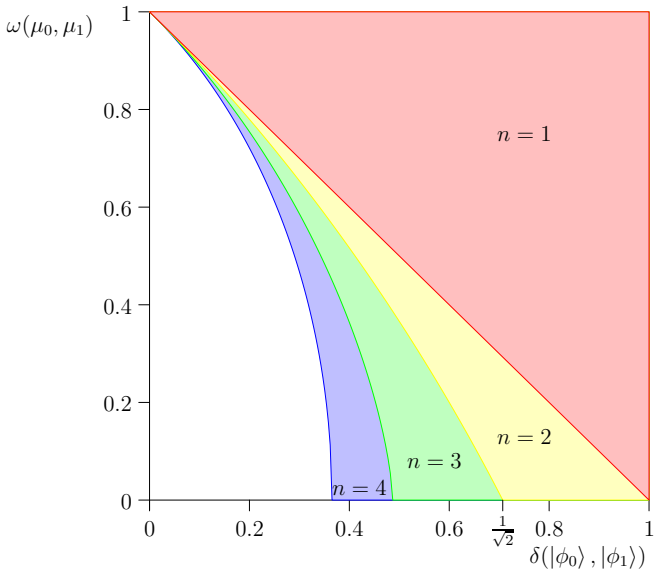

FIG. 4. The overlap $\omega(\mu_0, \mu_1)$ (Equation (B4)), versus the quantum trace distance $\delta(|\phi_0\rangle, |\phi_1\rangle) = \sqrt{1 - |\langle\phi_1 | \phi_0\rangle|^2}$. The red region is ruled out by measurements on a single system. The other regions can be ruled out by measurements on 2, 3 and 4 systems. The content of the no-go theorem is that larger and larger n eventually fill the square, forcing $\omega(\mu_0, \mu_1) = 0$ for any pair of states. The boundaries of the regions are not ruled out (except that $\omega(\mu_0, \mu_1) > 0$ *is* ruled out for $\delta(|\phi_0\rangle, |\phi_1\rangle) = 1$).

Appendix C: Numerical results

For a given $|\phi_0\rangle$ and $|\phi_1\rangle$, the measurement described in Appendix A requires n systems, with n such that

$$2 \arctan\left(2^{1/n} - 1\right) \leq \arccos |\langle\phi_0 | \phi_1\rangle| . \tag{C1}$$

It is natural to ask if there exists a measurement that can make do with smaller values of n. We have checked by numerically solving [15, 16] the semi-definite program

$$\begin{aligned} \underset{E_i}{\text{minimize}} \quad & \sigma := \sum_{\vec{x}} \mathrm{Tr}(E_{\vec{x}} |\Psi(\vec{x})\rangle \langle\Psi(\vec{x})|) \\ \text{subject to} \quad & E_{\vec{x}} \succeq 0, \\ & \sum_{\vec{x}} E_{\vec{x}} = \mathbb{I}. \end{aligned} \tag{C2}$$

Since all the terms in the definition of σ are non-negative, a measurement described by the POVM operators $\{E_{\vec{x}}\}$ can be used to prove the no-go theorem if and only if $\sigma = 0$. A variety of values of θ and n were tested, and the minimum value of σ was found to be 0 exactly when (C1) is satisfied. Hence it appears that our measurement uses the smallest possible number of systems.

Furthermore, when Eq. (C1) is not satisfied the optimal measurement is of same form, but with $\alpha = \pi$ and $\beta = 0$. (For $n = 1$ this measurement is simply the standard minimum error discrimination measurement for $|\phi_0\rangle$ and $|\phi_1\rangle$.) By a similar argument to the previous section, if the quantum theory predictions for this measurement hold, then $(\omega(\mu_0, \mu_1))^n \leq \sigma$. Hence, in addition to our main result that there exists a measurement showing $\omega(\mu_0, \mu_1) = 0$ when (C1) is satisfied, this measurement can be used to place bounds on $\omega(\mu_0, \mu_1)$ when it is not. The situation is depicted in Figure 4.

Finally, we note that the problem (C2) has an unusual operational interpretation. By considering each outcome $E_{\vec{x}}$ as the identification of $|\Psi(\vec{x})\rangle$, we have an error probability of $1 - \sigma/2^n$, and so this is the "maximum error" discrimination problem for the quantum states $\{|\Psi(\vec{x})\rangle\}$ (with equal priors). For the special cases of two states this becomes the minimum error problem under swapping of the outcome labels.

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